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Asymptotic behaviour of fundamental cycle of periodic box–ball systems

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Abstract

We investigate asymptotic behaviour of fundamental cycle of periodic box–ball systems (PBBSs) when the system size N goes to infinity. According to integrable nature of the PBBS, the trajectory is confined to qualitatively smaller number of states than that of the total states. We prove that, although the maximum fundamental cycle is of order of $\exp[\sqrt{N}]$, almost all fundamental cycle is less than $\exp[(\log N)^2]$.

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1. Introduction

The periodic box–ball system (PBBS) is a dynamical system of balls in a one-dimensional array of boxes with periodic boundary condition [1, 2]. The PBBS is obtained from the discrete Toda equation [3], which is a well-known integrable partial difference equation, with a periodic boundary condition through a limiting procedure called ultradiscretization [4, 5]. Using inverse ultradiscretization, the initial value problem of PBBS is solvable by inverse scattering transform [6]. Hence, the PBBS may be called an *integrable* dynamical system. On the other hand, an important feature of an integrable dynamical system is that its trajectory in the phase space is restricted to a low-dimensional subspace determined by the conserved quantities [7]. In particular, it does not have ergodicity on the phase space determined by the total energy (the energy surface). Accordingly, to see the qualitative difference between a trajectory of an integrable dynamical system and that of a non-integrable system, one may take the Poincaré section in the lower-dimensional plane. If we plot a two-dimensional Poincaré section for an integrable system, its trajectory locates on lower-dimensional curves and is quite different from that of non-integrable (or chaotic) systems. However, since the PBBS is composed of a finite number of boxes and balls, it can only take on a finite number of patterns. In other words, the phase space of the PBBS consists of only a finite number of points. For dynamical systems with such phase spaces, the difference between integrable and non-integrable systems cannot be clearly specified from the trajectory.

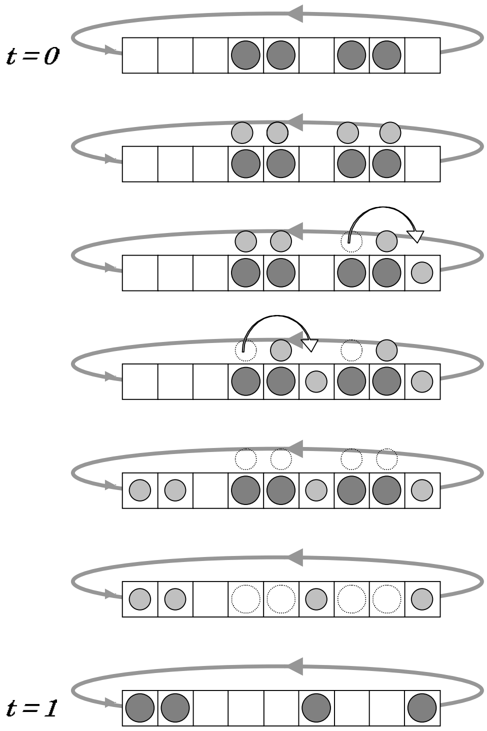

Figure 1. Time evolution rule for PBBS.

Recently Yoshihara *et al* have obtained the formulae to determine the fundamental cycle, i.e., the shortest period of the discrete periodic motion of the PBBS [8]. In the present paper, we examine the *integrability* of the PBBS from its fundamental cycle based on their results. Our point of view is quite naive—if a dynamical system has ergodicity in some sense, its trajectory starting from a generic point will cover a significant portion of the phase space, and the fundamental cycle T is of order of the volume (number of points) of the phase space. In contrast, if a dynamical system has integrability in some sense, T will be qualitatively smaller. We shall show that, in fact, a fundamental cycle of PBBS is much smaller than the volume of its phase space.

In section 2, we briefly give the definition of PBBS and summarize the results obtained in [8]. Using these results, we give upper and lower bounds of the fundamental cycle with respect to the system size N in section 3. The distribution function for the number of states with respect to the maximum length of solitons are evaluated in section 4 by means of a generating function. We can conclude that almost all initial states have the fundamental cycle less than $\exp[(\log N)^2]$ using the distribution function. Section 5 is devoted to concluding remarks.

2. Periodic box–ball system and its fundamental cycle

First we briefly summarize the results in [8]. Let us consider a one-dimensional array of N boxes. To be able to impose a periodic boundary condition, we assume that the Nth box is the adjacent box to the first one. The box capacity is one for all the boxes, and each box is either empty or filled with a ball at any time step. We denote the number of balls by M, such that $M \leqslant \frac{N}{2}$. The balls are moved according to a deterministic time evolution rule (figure 1).

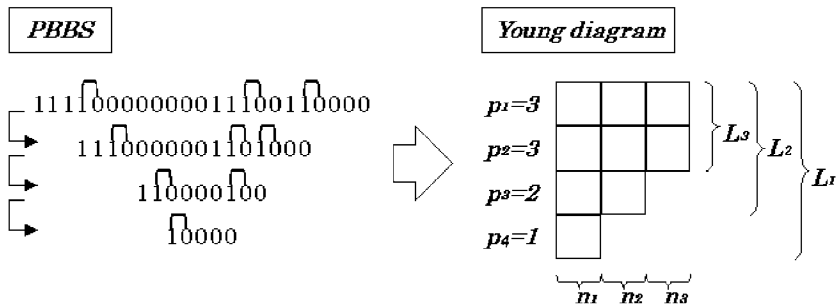

Figure 2. Correspondence of PBBS and Young diagram.

1. In each filled box, create a copy of the ball.
2. Move all the copies once according to the following rules.
3. Choose one of the copies and move it to the nearest empty box on the right of it.
4. Choose one of the remaining copies and move it to the nearest empty box on the right of it.
5. Repeat the above procedure until all the copies have moved.
6. Delete all the original balls.

A PBBS has conserved quantities, which are characterized by a Young diagram with M boxes (figure 2). The Young diagram is constructed as follows. We denote an empty box by '0' and a filled box by '1'. Then the PBBS is represented as a 0, 1 sequence in which the last entry is regarded as adjacent to the first entry. Let p_1 be the number of 10 pairs in the sequence. If we eliminate these 10 pairs, we obtain a new 0, 1 sequence. We denote by p_2 the number of 10 pairs in the new sequence. We repeat the above procedure until all the '1's are eliminated and obtain $p_2, p_3, \ldots, p_l$. Clearly $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_l$ and $\sum_{i=1}^{l} p_i = M$. These $\{p_i\}_{i=1}^{l}$ are conserved in time evolution. Since $\{p_1, p_2, \ldots, p_l\}$ is a weakly decreasing series of positive integers, we can associate it with a Young diagram with p_j boxes in the jth column ($j = 1, 2, \ldots, l$). Then the lengths of the rows are also weakly decreasing positive integers, and we denote them

$$\{\underbrace{L_1, L_1, \ldots, L_1}_{n_1}, \underbrace{L_2, L_2, \ldots, L_2}_{n_2}, \ldots, \underbrace{L_s, L_s, \ldots, L_s}_{n_s}\}$$

where $L_1 > L_2 > \cdots > L_s$. The set $\{L_j, n_j\}_{j=1}^{s}$ is an alternative expression of the conserved quantities of the system. In the limit $N \to \infty$, L_j means the length of the jth largest soliton and n_j is the number of solitons with length L_j.

The following two propositions are essential in our arguments. Let $\ell_0 := N - 2M = N - \sum_{j=1}^{l} 2p_j = N - \sum_{j=1}^{s} 2n_j L_j$, $N_0 := \ell_0$, $L_{s+1} := 0$ and

$$\ell_j := L_j - L_{j+1} \qquad (j = 1, 2, \ldots, s) \tag{2.1}$$

$$\begin{aligned} N_j &:= \ell_0 + 2n_1(L_1 - L_{j+1}) + 2n_2(L_2 - L_{j+1}) + \cdots + 2n_j(L_j - L_{j+1}) \\ &= \ell_0 + \sum_{k=1}^{j} 2n_k(L_k - L_{j+1}) \qquad (j = 1, 2, \ldots, s). \end{aligned} \tag{2.2}$$

Then, for a fixed number of boxes N and conserved quantities $\{L_j, n_j\}$, the number of possible states of the PBBS $\Omega(N; \{L_j, n_j\})$ is given by the following formula.

Proposition 2.1 (YYT)**.**

$$\Omega(N;\{L_j,n_j\}) = \frac{N}{\ell_0}\binom{\ell_0+n_1-1}{n_1}\binom{N_1+n_2-1}{n_2}\binom{N_2+n_3-1}{n_3}\times\cdots\times\binom{N_{s-1}+n_s-1}{n_s}. \tag{2.3}$$

Note that the formula (2.3) holds when some of the n_j are equal to 0, i.e.,

$$\Omega\big(N;\{L_j,n_j\}_{j=1}^{s}\big) = \Omega\big(N;\{i,n_i\}_{i=1}^{M}\big) \tag{2.4}$$

where $n_i = 0$ if $i \notin \{L_j\}_{j=1}^{s}$ and $n_i = n_j$ if $L_j = i$.

The fundamental cycle T is given as

Proposition 2.2 (YYT)**.** *Let $\tilde{T}$ be defined as*

$$\tilde{T} := \mathrm{LCM}\left(\frac{N_s N_{s-1}}{\ell_s\ell_0}, \frac{N_{s-1}N_{s-2}}{\ell_{s-1}\ell_0}, \ldots, \frac{N_1 N_0}{\ell_1\ell_0}, 1\right) \tag{2.5}$$

where $\mathrm{LCM}(x,y) := 2^{\max[x_2,y_2]}3^{\max[x_3,y_3]}5^{\max[x_5,y_5]}\cdots$ *for* $x = 2^{x_2}3^{x_3}5^{x_5}\cdots$ *and* $y = 2^{y_2}3^{y_3}5^{y_5}\cdots$. *Then T is a divisor of $\tilde{T}$. In particular, when there is no internal symmetry in the state $T = \tilde{T}$.*

The definition of internal symmetry in the above proposition is rather complicated and we refer to the original paper [8]. However, for a given number of conserved quantities, we can always construct initial states, which do not have any internal symmetry, in particular, if $^{\forall}i, n_i = 1$ the PBBS has no internal symmetry and $T = \tilde{T}$.

3. Maximum value of the fundamental cycle

To take an appropriate limit, we fix the ball density $\rho := M/N$. The volume of the phase space $V(N;\rho)$ is

$$V(N;\rho) = \binom{N}{M} \sim \frac{1}{\sqrt{2\pi\rho(1-\rho)N}}R^N \qquad (R := (1-\rho)^{\rho-1}\rho^{-\rho}). \tag{3.1}$$

Thus the volume of the phase space increases exponentially with respect to the system size N. On the other hand, for a given number of balls M, there are P_M different Young diagrams which correspond to conserved quantities. Here P_M is the number of partitions of M. The following estimation of P_M is well known [9].

$$P_M = \frac{\exp[\pi\sqrt{2M/3}]}{4\sqrt{3}M}\left(1 + O\left(\frac{\log M}{M^{1/4}}\right)\right). \tag{3.2}$$

Since $M = \rho N$, we have $P_M \sim \exp[\pi\sqrt{2\rho/3}\sqrt{N}]/(4\sqrt{3}\rho N)$. The restricted phase space determined by the conserved quantities has the volume $V(N;\rho)/P_M$ on average. This average volume still grows exponentially with respect to the system size. Since we can only say that a fundamental cycle of the PBBS is at most less than the volume of the subspace determined by these conserved quantities, we need more detailed analysis to know the asymptotic behaviour of the fundamental cycle of PBBS.

In this section, we estimate the maximum fundamental cycle $T_{\max} := \max[T]$. From (2.5) $\tilde{T}$ is evaluated as

$$\begin{aligned}\tilde{T} &\leqslant \mathrm{LCM}(N_s N_{s-1}, N_{s-1}N_{s-2}, \ldots, N_2 N_1)\\ &\leqslant \prod_{j=1}^{s} N_j < (N_s)^s = N^s.\end{aligned}$$

Since

$$M = \sum_{j=1}^{s} n_j L_j \geqslant \sum_{j=1}^{s} j = \frac{s(s+1)}{2}$$

we find $s < \sqrt{2M}$ and

$$N^s = \mathrm{e}^{s \log N} < \mathrm{e}^{\sqrt{2\rho N} \log N}.$$

Thus we have an upper bound

$$T_{\max} < \mathrm{e}^{\sqrt{2\rho}\sqrt{N} \log N}. \tag{3.3}$$

Next we estimate a lower bound of $T_{\max}$. First we assume N is an even integer. Since $\ell_0 = N - 2M$, ℓ_0 is also an even integer. Let k and s be the integers which are determined uniquely by

$$k(k-1) \leqslant \ell_0 \leqslant k(k+1) - 2 \tag{3.4}$$

$$(k+s-1)(k+s) \leqslant N < (k+s)(k+s+1). \tag{3.5}$$

Then we consider an initial state which consists of s kinds of solitons with length $1, 2, \ldots, s$ ($\ell_j = 1 \forall j \geqslant 1$). From (3.4) and (3.5), we may take $n_1 = \frac{k(k+1)-\ell_0}{2}$, $n_2 = \frac{\ell_0 - k(k-1)+2}{2}$, $n_s = \frac{N-(k+s)(k+s-1)+2}{2}$ and $n_j = 1$ $(3 \leqslant j \leqslant s-1)$. By the definition of N_j (2.2), we have

$$N_1 = k(k+1) \qquad N_2 = (k+1)(k+2) \qquad N_3 = (k+2)(k+3), \ldots,$$
$$N_{s-1} = (k+s-2)(k+s-1) \quad (N_s \equiv N).$$

As was mentioned in the previous section, we can suppose that there is no internal symmetry in this state and its fundamental cycle $T^{(k)}$ is estimated as

$$\begin{aligned}
T^{(k)} &= \mathrm{LCM}\left(\frac{N_s N_{s-1}}{\ell_s \ell_0}, \frac{N_{s-1} N_{s-2}}{\ell_{s-1}\ell_0}, \ldots, \frac{N_2 N_1}{\ell_1 \ell_0}\right) \\
&\geqslant \frac{1}{\ell_0} \mathrm{LCM}(N_{s-1}N_{s-2}, N_{s-2}N_{s-3}, \ldots, N_2 N_1) \\
&= \frac{1}{\ell_0} \mathrm{LCM}((k+s-1)(k+s-2)^2(k+s-3), \ldots, (k+3)(k+2)^2(k+1), (k+2)(k+1)^2 k) \\
&\geqslant \frac{1}{\ell_0} \mathrm{LCM}((k+s-2)^2, (k+s-3)^2, \ldots, (k+1)^2) \\
&= \frac{1}{\ell_0} (\mathrm{LCM}((k+s-2), (k+s-3), \ldots, (k+1)))^2.
\end{aligned} \tag{3.6}$$

We define

$$L(n, m) := \mathrm{LCM}(n, n-1, \ldots, m+2, m+1) \tag{3.7}$$

for positive integer n and m $(n > m)$, then the right-hand side of (3.6) is rewritten as $L(k+s-2, k)^2$. From the identities $\mathrm{LCM}(A, B) = AB/\mathrm{GCD}(A, B)$ and $L(n, 1) = \mathrm{LCM}(L(n, m), L(m, 1))$, we know

$$L(n, m) = L(n, 1) \frac{\mathrm{GCD}(L(n, m), L(m, 1))}{L(m, 1)}.$$

Since $L(n-m, 1)$ is a divisor of $L(n, m)$, $\mathrm{GCD}(L(n, m), L(m, 1)) \geqslant \mathrm{GCD}(L(n-m, 1), L(m, 1)) \geqslant \min[L(n-m, 1), L(m, 1)]$ and $L(n, m)$ satisfies

$$L(n, m) \geqslant \frac{L(n, 1) \min[L(n-m, 1), L(m, 1)]}{L(m, 1)}.$$

If we introduce the Chebyshev function $\psi(n)$ [9]

$$\psi(n) := \sum_{p^j \leqslant n,\, p \text{ is a prime},\, j \in \mathbb{Z}_+} \log p \tag{3.8}$$

$L(n,1) = \mathrm{LCM}(n, n-1, n-2, \ldots, 2)$ is expressed as

$$L(n,1) = \exp[\psi(n)] \tag{3.9}$$

and we obtain the inequality

$$L(n,m) \geqslant \exp[\psi(n) - \max[\psi(m) - \psi(n-m), 0]]. \tag{3.10}$$

The asymptotic formulae for the Chebyshev function $\psi(n)$ have been extensively investigated since the nineteenth century [10]. For example, the following lemma holds [11]:

Lemma 3.1 (Rooser–Schoenfeld)**.** *If* $n \geqslant 10^8$ *then*

$$|\psi(n) - n| < 0.025 \frac{n}{\log n}. \tag{3.11}$$

Thus we have rather a rough estimation

$$L(n,m) > \exp\left[(n - \max[2m-n, 0])\left(1 - \frac{c}{\log n}\right)\right] \qquad \text{for} \quad n, m \gg 1 \tag{3.12}$$

where c is a small positive number. (From lemma 3.1, we can take $c \sim 0.1$ for $N \geqslant 10^{16}$.) Using this inequality and (3.6), the fundamental cycle $T^{(k)}$ is estimated as

$$T^{(k)} > \exp\left[2((k+s-2) - \max[k-s+2, 0])\left(1 - \frac{c}{\log(k+s-2)}\right)\right]$$

for $k \gg 1$. From (3.4) and (3.5), we have $\sqrt{N} - 1 < k + s < \sqrt{N} + 1$ and $\sqrt{\ell_0} - 1 < k < \sqrt{\ell_0} + 1$. Therefore, we obtain

$$T^{(k)} > \exp\left[2(1 - \max[\sqrt{2-4\rho} - 1, 0])\sqrt{N}\left(1 - \frac{c}{\log N}\right)\right] \qquad \text{for} \quad N \gg 1. \tag{3.13}$$

In the case N is an odd integer, ℓ_0 is also an odd integer and we determine k (odd number) and s by

$$k(k-2) \leqslant \ell_0 \leqslant k(k+2) - 2 \tag{3.14}$$

$$(k+2s-2)(k+2s) \leqslant N < (k+2s)(k+2s+2). \tag{3.15}$$

Then we again consider an initial state which consists of s kinds of solitons with length $1, 2, \ldots, s$ ($\ell_j = 1 \forall j \geqslant 1$). From (3.14) and (3.15), we may take $n_1 = \frac{k(k+2)-\ell_0}{2}$, $n_2 = \frac{\ell_0 - k(k-2)+8}{2}$, $n_s = \frac{N-(k+2s)(k+2s-2)+8}{2}$ and $n_j = 4$ $(3 \leqslant j \leqslant s-1)$. For N_j, we have

$$N_1 = k(k+2) \qquad N_2 = (k+2)(k+4) \qquad N_3 = (k+4)(k+6), \ldots,$$
$$N_{s-1} = (k+2s-4)(k+2s-2) \quad (N_s \equiv N).$$

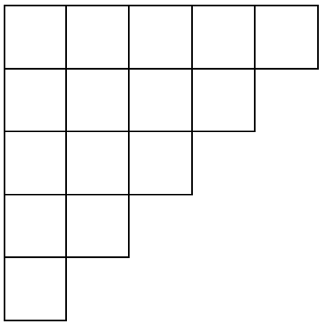

Figure 3. An example of a triangular Young diagram.

The fundamental cycle $T'^{(k)}$ is estimated in a similar manner to the even integer case as

$$T'^{(k)} \geqslant \frac{1}{\ell_0}(\mathrm{LCM}((k+2s-2),(k+2s-4),\ldots,(k+2)))^2. \tag{3.16}$$

Thus we can again use the asymptotic formulae for the Chebyshev function and obtain

$$T'^{(k)} > \exp\left[2(1-\max[\sqrt{2-4\rho}-1,0])\sqrt{N}\left(1-\frac{c}{\log N}\right)\right] \qquad \text{for} \quad N \gg 1. \tag{3.17}$$

Therefore, we have proved the following theorem:

Theorem 3.1. *For $N \gg 1$ and $M = \rho N$ $(0 < \rho < 1/2)$, the maximum value of the fundamental cycle $T_{\max} \equiv T_{\max}(N;\rho)$ satisfies*

$$\exp\left[2(1-\max[\sqrt{2-4\rho}-1,0])\sqrt{N}\left(1-\frac{c}{\log N}\right)\right] < T_{\max} < \exp[\sqrt{2\rho}\sqrt{N}\log N]. \tag{3.18}$$

Here c is a positive integer and $c \sim 0.1$ for $N \geqslant 10^{16}$.

From theorem 3.1, we find that $\log T(N;\rho) \lesssim \sqrt{N}$. On the other hand $\log V(N;\rho) \sim N$, and we may be able to conclude that the PBBS does not have the ergodic property in the sense given in section 1.

Although formula (3.18) is a rather rough estimation for the maximum fundamental cycle, it seems a difficult problem to obtain a sharper bound for $T_{\max}$ analytically because of its number theoretical aspects. From the above arguments and numerical calculations, however, we expect that the fundamental cycle of the initial state, which has the conserved quantities determined by the triangular Young diagram (see figure 3) for the partition $(s, s-1, s-2, \ldots, 2, 1)$, is almost of the order of $T_{\max}$. In this case, all the solitons have different lengths and the fundamental cycle is given as

$$T^{(t)}(N,\rho) = \mathrm{LCM}\left(\frac{N_s N_{s-1}}{\ell_0}, \frac{N_{s-1}N_{s-2}}{\ell_0}, \ldots, \frac{N_1 N_0}{\ell_0}, 1\right) \tag{3.19}$$

where $N_k = \ell_0 + k(k+1)$ and $\ell_0 = N - 2M = (\rho^{-1}-2)s(s+1)/2$.

The number of possible states for the triangular Young diagram $\Omega^{(t)}(N,\rho)$ is given as

$$\Omega^{(t)}(N,\rho) = \prod_{k=1}^{s}(\ell_0 + k(k+1)) \tag{3.20}$$

where $M = \rho N = s(s+1)/2$ and $\ell_0 = (1-2\rho)N$. By putting $\gamma := \ell_0/s^2$, we have

$$\begin{aligned}
\Omega^{(t)}(N,\rho) &= s^{2s}\prod_{k=1}^{s}\left[\gamma + \left(\frac{k}{s}\right)\left(\frac{k+1}{s}\right)\right] \\
&= s^{2s}\exp\left[\sum_{k=1}^{s}\log\left[\gamma + \left(\frac{k}{s}\right)\left(\frac{k+1}{s}\right)\right]\right] \\
&\simeq s^{2s}\exp\left[s\int_0^1 \log(\gamma + x^2)\,\mathrm{d}x\right] \\
&= s^{2s}\exp\left[s\left(\log(1+\gamma) - 2 + 2\sqrt{\gamma}\arctan\frac{1}{\sqrt{\gamma}}\right)\right].
\end{aligned}$$

Since $\gamma = -1 + 1/(2\rho)$, by putting $\alpha(\rho) := \log(1+\gamma) - 2 + 2\sqrt{\gamma}\arctan\frac{1}{\sqrt{\gamma}} + \log(2\rho)$, we have

$$\Omega^{(t)}(N,\rho) \simeq \exp[\sqrt{2\rho}\sqrt{N}(\log N + \alpha(\rho))]. \tag{3.21}$$

Thus $\Omega^{(t)}(N,\rho) \sim \mathrm{e}^{(\sqrt{2\rho})\sqrt{N}\log N}$ and

$$\frac{\log\Omega^{(t)}(N,\rho)}{\log V(N,\rho)} \sim \frac{\log N}{\sqrt{N}}.$$

Hence the number of possible states for the triangular Young diagram is much smaller than the volume of the phase space. Figure 4 shows the ratio $T^{(t)}(N,\rho)/\Omega^{(t)}(N,\rho)$ obtained numerically. The results show that the fundamental cycle $T^{(t)}$ is much smaller than the number of states $\Omega^{(t)}$. Although the results are not enough to estimate the asymptotic value of $T^{(t)}$, we see in this example that, even if we restrict ourselves to the phase space determined by the conserved quantities, the trajectory does not have ergodicity in the sense that it will never visit most of the states with the same conserved quantities.

4. Asymptotic behaviour of fundamental cycle for generic initial states

In the preceding section, we have proved that $\log T_{\max} \sim \sqrt{N}$. For a generic initial state, however, we expect that its fundamental a cycle is qualitatively much smaller. For example, initial states correspond to a rectangular Young diagram (figure 5). In this case, its fundamental cycle is easily obtained as a divisor of $\tilde{T}^{(r)}(N)$:

$$\tilde{T}^{(r)}(N) = \mathrm{LCM}\left(\frac{N}{L_1}, 1\right) \leqslant N.$$

Hence the fundamental cycle less than or equal to the system size N. In the case of figure 5(b) ($L_1 = 1, n_1 = M$), the number of possible states $\Omega^{(r_b)}(N,\rho)$ for the rectangular Young diagram (b) is given as

$$\begin{aligned}
\Omega^{(r_b)}(N,\rho) &= \frac{N}{N-2M}\binom{N-M-1}{M} \\
&\sim \frac{1}{\sqrt{2\pi\rho(1-\rho)(1-2\rho)N}}\tilde{R}^N \qquad (\tilde{R} := (1-\rho)^{1-\rho}\rho^{-\rho}(1-2\rho)^{2\rho-1}).
\end{aligned}$$

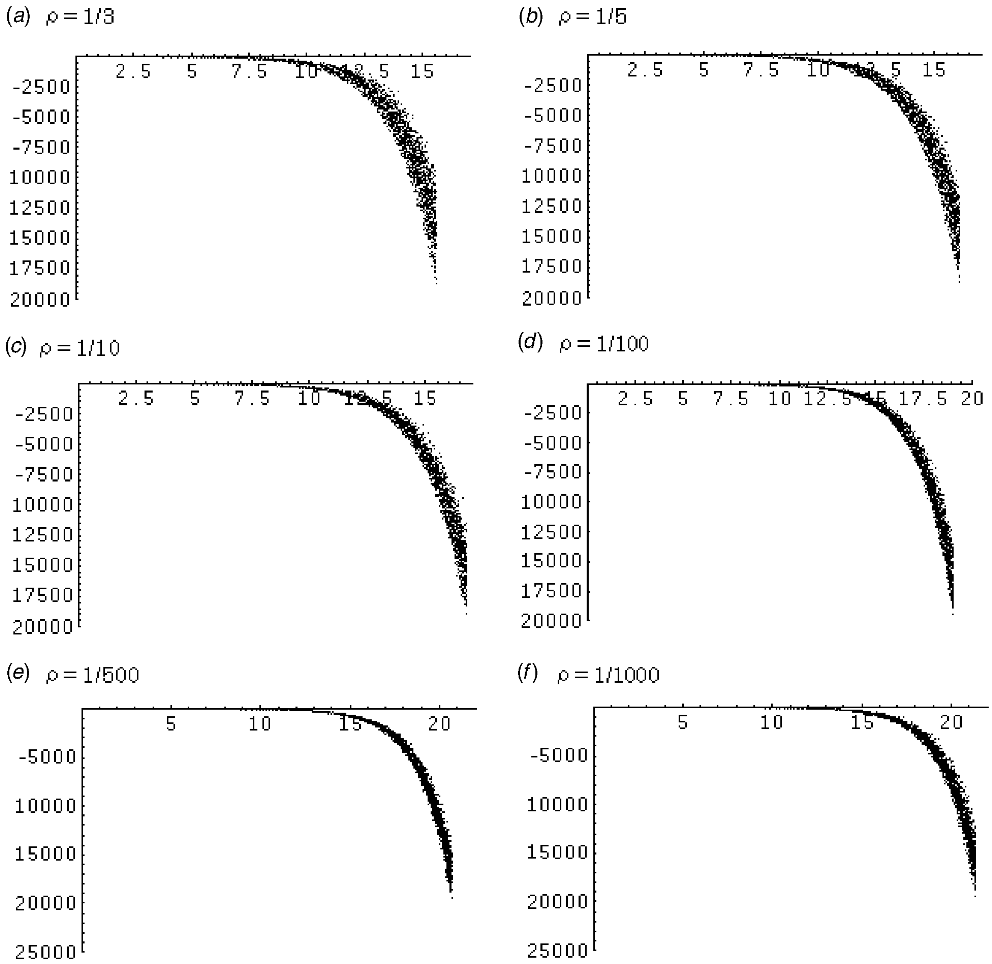

Figure 4. Results of numerically calculated $\log[T^{(t)}(N, \rho)/\Omega^{(t)}(N, \rho)]$.

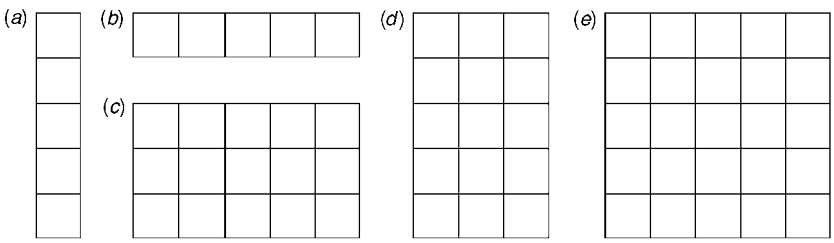

Figure 5. Rectangular Young diagram.

Therefore, the number of these initial states grows exponentially with respect to N, while that of the initial states corresponding to triangular diagrams grows much more slowly like (3.21).

To examine the asymptotic behaviour for a generic initial state, we define the generating function as

$$
\begin{aligned}
F(N, K, \ell_0; x) &:= \left(\prod_{j=1}^{K} \sum_{n_j=0}^{\infty}\right) \Omega(N; \{j, n_j\}) x^{\sum_{i=1}^{K} i n_i} \\
&= \frac{N}{\ell_0} \left(\prod_{j=1}^{K} \sum_{n_j=0}^{\infty}\right) \binom{\ell_0 + n_K - 1}{n_K} \\
&\quad \times \binom{\ell_0 + 2n_K + n_{K-1} - 1}{n_{K-1}} \binom{\ell_0 + 4n_K + 2n_{K-1} + n_{K-2} - 1}{n_{K-2}} \\
&\quad \times \cdots \times \binom{\ell_0 + \left(\sum_{i=2}^{K} 2(i-1)n_i\right) + n_1 - 1}{n_1} x^{\sum_{i=1}^{K} i n_i}.
\end{aligned}
\tag{4.1}
$$

From proposition 2.1 and equation (2.4), we find

Proposition 4.1. *Let N, M and ℓ_0 be the number of boxes of a PBBS, the number of balls and $\ell_0 = N - 2M$, respectively. Then the coefficient of x^M of $F(N, K, \ell_0; x)$ is the number of initial states whose largest solitons have length less than or equal to K.*

The function $F(N, K, \ell_0; x)$ has the following expression:

Proposition 4.2.

$$
F(N, K, \ell_0; x) = \frac{N}{\ell_0} (Y_K(x))^{\ell_0} \tag{4.2}
$$

where $Y_K(x)$ is recursively defined as

$$
\begin{aligned}
X_1(x) &:= \frac{1}{1-x} \\
Y_k(x) &:= X_1(x) X_2(x) \cdots X_k(x) \qquad (k = 1, 2, \ldots) \\
X_k(x) &:= \frac{1}{1 - \{Y_1(x) Y_2(x) \cdots Y_{k-1}(x)\}^2 x^k} \qquad (k = 1, 2, \ldots).
\end{aligned}
\tag{4.3}
$$

Proof. From the identity

$$
\sum_{n=0}^{\infty} \binom{a+n-1}{n} x^n = \frac{1}{(1-x)^a} \tag{4.4}
$$

we have

$$
\begin{aligned}
F(N, K, \ell_0; x) &= \frac{N}{\ell_0} \sum_{n_j \geqslant 0, j \neq 1} \frac{1}{(1-x)^{\ell_0 + 2(K-1)n_K + \cdots + 2n_2}} \binom{\ell_0 + n_K - 1}{n_K} \\
&\quad \times \cdots \times \binom{\ell_0 + 2(K-2)n_K + \cdots + 2n_3 + n_2 - 1}{n_2} (x^K)^{n_K} \cdots (x^2)^{n_2} \\
&= \frac{N}{\ell_0} \sum_{n_j \geqslant 0, j \neq 1} (X_1(x))^{\ell_0 + 2(K-1)n_K + \cdots + 4n_3} \\
&\quad \times \binom{\ell_0 + n_K - 1}{n_K} \cdots \binom{\ell_0 + 2(K-2)n_K + \cdots + 2n_3 + n_2 - 1}{n_2} \\
&\quad \times (x^K)^{n_K} \cdots (x^3)^{n_3} ((X_1(x))^2 x^2)^{n_2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{N}{\ell_0} \sum_{n_j \geqslant 0,\, j \neq 1} (X_1(x))^{\ell_0 + 2(K-1)n_K + \cdots + 4n_3} \\
&\quad \times \binom{\ell_0 + n_K - 1}{n_K} \cdots \binom{\ell_0 + 2(K-2)n_K + \cdots + 2n_3 + n_2 - 1}{n_2} \\
&\quad \times (x^K)^{n_K} \cdots (x^3)^{n_3} ((Y_1(x))^2 x^2)^{n_2}.
\end{aligned}
\tag{4.5}
$$

Repeated use of (4.4) and the definition of $X_k(x)$ and $Y_k(x)$ yields

$$
\begin{aligned}
F(N, K, \ell_0; x) &= \frac{N}{\ell_0} \sum_{n_j \geqslant 0,\, j \neq 1,2} (X_1(x))^{\ell_0 + 2(K-1)n_K + \cdots + 4n_3} \\
&\quad \times \frac{1}{(1 - (Y_1(x))^2 x^2)^{\ell_0 + 2(K-2)n_K + \cdots + 2n_3}} \binom{\ell_0 + n_K - 1}{n_K} \\
&\quad \times \cdots \times \binom{\ell_0 + 2(K-3)n_K + \cdots + 2n_4 + n_3 - 1}{n_3} (x^K)^{n_K} \cdots (x^3)^{n_3} \\
&= \frac{N}{\ell_0} \sum_{n_j \geqslant 0,\, j \neq 1,2} (X_1(x))^{\ell_0 + 2(K-1)n_K + \cdots + 8n_4} (X_2(x))^{\ell_0 + 2(K-2)n_K + \cdots + 4n_4} \\
&\quad \times \binom{\ell_0 + n_K - 1}{n_K} \cdots \binom{\ell_0 + 2(K-3)n_K + \cdots + 2n_4 + n_3 - 1}{n_3} \\
&\quad \times (x^K)^{n_K} \cdots (x^3)^{n_3} \{(X_1(x) X_1(x) X_2(x))^2 x^3\}^{n_3} \\
&= \frac{N}{\ell_0} \sum_{n_j \geqslant 0,\, j \neq 1,2} (X_1(x))^{\ell_0 + 2(K-1)n_K + \cdots + 8n_4} (X_2(x))^{\ell_0 + 2(K-2)n_K + \cdots + 4n_4} \\
&\quad \times \binom{\ell_0 + n_K - 1}{n_K} \cdots \binom{\ell_0 + 2(K-3)n_K + \cdots + 2n_4 + n_3 - 1}{n_3} \\
&\quad \times (x^K)^{n_K} \cdots (x^3)^{n_3} \{(Y_1(x) Y_2(x))^2 x^3\}^{n_3} \\
&= \cdots \\
&= \frac{N}{\ell_0} (X_1(x))^{\ell_0} (X_2(x))^{\ell_0} \cdots (X_K(x))^{\ell_0} \\
&= \frac{N}{\ell_0} (Y_K(x))^{\ell_0}.
\end{aligned}
$$

□

Now we introduce

$$
a_k(x) := \sum_{j=0}^{[\frac{k+1}{2}]} \binom{k+1-j}{j} (-1)^j x^j \qquad (k \geqslant -1, k \in \mathbb{Z}). \tag{4.6}
$$

For polynomials $a_k(x)$, we have the following lemma.

Lemma 4.1. *Let $a_k(x)$ be as above, then*

$$
a_{k+1}(x) = a_k(x) - x a_{k-1}(x) \qquad (k = 0, 1, 2, \ldots) \tag{4.7}
$$

$$
a_{k+1}(x) a_{k-1}(x) = a_k(x)^2 - x^{k+1} \qquad (k = 0, 1, 2, \ldots) \tag{4.8}
$$

$$
a_k(x) = \frac{\alpha(x)^{k+2} - \beta(x)^{k+2}}{\alpha(x) - \beta(x)} \qquad (k = 0, 1, 2, \ldots) \tag{4.9}
$$

where $\alpha(x)$ and $\beta(x)$ are two distinct roots of the quadratic equation

$$
t^2 - t + x = 0.
$$

Note that $\alpha(x)$ and $\beta(x)$ are explicitly given as

$$\alpha(x) = \frac{1+\sqrt{1-4x}}{2} \qquad \beta(x) = \frac{1-\sqrt{1-4x}}{2} \tag{4.10}$$

and $\alpha(x)\beta(x) = x$, $\alpha(x)+\beta(x) = 1$.

Proposition 4.3.

$$Y_k(x) = \frac{a_{k-1}(x)}{a_k(x)} \tag{4.11}$$

$$= \frac{\alpha(x)^{k+1} - \beta(x)^{k+1}}{\alpha(x)^{k+2} - \beta(x)^{k+2}} \quad (k = 1, 2, 3, \ldots). \tag{4.12}$$

Proof. Since (4.12) is easily seen from (4.9) and (4.11), we prove (4.11) by induction. Since $a_0(x) = 1, a_1(x) = 1 - x$, we have $Y_1(x) = \frac{a_0(x)}{a_1(x)}$. Suppose that (4.11) holds for $k = 1, 2, \ldots, n$. By the definition of $X_k(x)$ (4.3), we have

$$\begin{aligned} X_{n+1}(x) &= \frac{1}{1 - \{Y_1(x)Y_2(x)\cdots Y_n(x)\}^2 x^{n+1}} \\ &= \frac{1}{1 - \left(\frac{a_0(x)}{a_1(x)}\frac{a_1(x)}{a_2(x)}\cdots\frac{a_{n-1}(x)}{a_n(x)}\right)^2 x^{n+1}} \\ &= \frac{1}{1 - \left(\frac{a_0(x)}{a_n(x)}\right)^2 x^{n+1}} \\ &= \frac{(a_n(x))^2}{(a_n(x))^2 - x^{n+1}}. \end{aligned} \tag{4.13}$$

However, from (4.8),

$$X_{n+1}(x) = \frac{(a_n(x))^2}{(a_{n+1}(x)a_{n-1}(x))}.$$

Since $Y_{n+1}(x) = X_{n+1}(x)Y_n(x)$, (4.11) holds for $k = n + 1$. Hence (4.11) holds for $k = 1, 2, 3, \ldots$ by mathematical induction. □

From propositions 4.2 and 4.3, we have an explicit form of the generating function $F(N, K, \ell_0; x)$. Then the coefficient of x^M of $F(N, K, \ell_0; x)$, $f(N, K; M)$, is given by the contour integral

$$f(N, K; M) = \frac{1}{2\pi\mathrm{i}} \oint_{|z|=\epsilon\ll 1} \frac{F(N, K, \ell_0; z)}{z^{M+1}}\,\mathrm{d}z. \tag{4.14}$$

The asymptotic behaviour of the right-hand side of (4.14) may be estimated with, for example, the method of steepest decent. However, (4.14) is still complicated and we shall try to obtain a simpler expression.

The following lemma is easily obtained by induction.

Lemma 4.2.

$$\left(\frac{1}{\alpha(x)}\right)^m = \sum_{r=0}^{\infty} \frac{m(2r+m-1)!}{(r+m)!r!} x^r \qquad (m = 1, 2, \ldots). \tag{4.15}$$

Then we obtain an explicit formula for $f(N,K;M)$ defined in proposition 4.1 as

Proposition 4.4.

$$f(N,K;M) := \frac{N}{\ell_0}\sum_{j=0,(K+1)j+(K+2)i\leqslant M}^{\ell_0}\sum_{i=0}^{\infty}\binom{\ell_0}{j}\binom{\ell_0+i-1}{i}(-1)^j \times \frac{(\ell_0+2(K+1)j+2(K+2)i)(2M+\ell_0-1)!}{(M+\ell_0+(K+1)j+(K+2)i)!(M-(K+1)j-(K+2)i)!} \tag{4.16}$$

where $\ell_0 = N-2M$.

Proof. From (4.12), we have

$$Y_k(x) = \frac{1}{\alpha(x)}\frac{1-\left(\frac{\beta(x)}{\alpha(x)}\right)^{k+1}}{1-\left(\frac{\beta(x)}{\alpha(x)}\right)^{k+2}}. \tag{4.17}$$

Using (4.10), we know

$$Y(x) := \lim_{k\to+\infty} Y_k(x) = \frac{1}{\alpha(x)} \qquad \frac{\beta(x)}{\alpha(x)} = x(Y(x))^2.$$

Thus we find

$$(Y_k(x))^{\ell_0} = (Y(x))^{\ell_0}\left(\frac{1-(x(Y(x))^2)^{k+1}}{1-(x(Y(x))^2)^{k+2}}\right)^{\ell_0}$$
$$= \sum_{j=0}^{\ell_0}\sum_{i=0}^{\infty}\binom{\ell_0}{j}\binom{\ell_0+i-1}{i}(-1)^j x^{(k+1)j+(k+2)i}(Y(x))^{\ell_0+2(k+1)j+2(k+2)i}. \tag{4.18}$$

Since $Y(x)=\frac{1}{\alpha(x)}$, using lemma 4.2, we obtain a series expansion of $(Y_k(x))^{\ell_0}$ in terms of x which gives (4.16). $\square$

Equation (4.16) is rewritten as

$$f(N,K;M) = \frac{1}{\ell_0}\sum_{i=0,(K+1)j+(K+2)i\leqslant M}^{\infty}\binom{\ell_0+i-1}{i} \times \sum_{j=0}^{\ell_0}(-1)^j\binom{\ell_0}{j}\binom{2M+\ell_0}{M-(K+1)j-(K+2)i}(\ell_0+2(K+1)j+2(K+2)i).$$

Noticing the facts

$$\sum_{j=0}^{\ell_0}(-1)^j\binom{\ell_0}{j}\binom{2M'+\ell_0'}{M'-(k+1)j} = (1-x^{k+1})^{\ell_0}(1+x)^{2M'+\ell_0'}|_{x^{M'}}$$
$$\sum_{j=0}^{\ell_0}(-1)^j\binom{\ell_0}{j}\binom{2M'+\ell_0'}{M'-(k+1)j}j = -\ell_0 x^{k+1}(1-x^{k+1})^{\ell_0-1}(1+x)^{2M'+\ell_0'}|_{x^{M'}}$$

where $f(x)|_{x^{M'}}$ denotes the coefficient of $x^{M'}$ in the power series expansion of a function $f(x)$, we find

$$\sum_{j=0}^{\ell_0}(-1)^j\binom{\ell_0}{j}\binom{2M'+\ell_0'}{M'-(k+1)j}(\ell_0'+2(k+1)j)$$
$$= (\ell_0'(1-x^{k+1})-2(k+1)\ell_0 x^{k+1})(1-x^{k+1})^{\ell_0-1}(1+x)^{2M'+\ell_0'}|_{x^{M'}}.$$

By taking $\ell_0' = \ell_0 + 2(k+2)i$ and $M' = M - 2(k+1)i$, we have the following formula in a similar manner.

$$\sum_{i=0}^{\infty}\binom{\ell_0+i-1}{i}\sum_{j=0}^{\ell_0}(-1)^j\binom{\ell_0}{j}\binom{2M+\ell_0}{M-(k+1)j-(k+2)i}(\ell_0+2(k+1)j+2(k+2)i)$$
$$= (1-x^{k+1})^{\ell_0-1}(1+x)^{2M+\ell_0}\sum_{i=0}^{\infty}\binom{\ell_0+i-1}{i}x^{(k+2)i}$$
$$\times(2(k+2)(1-x^{k+1})i+\ell_0(1-(2k+3)x^{k+1}))|_{x^M}$$
$$= \frac{\ell_0(1-x^{k+1})^{\ell_0}(1+x)^{2M+\ell_0}}{(1-x^{k+2})^{\ell_0}}$$
$$\times\left(1-(2k+2)\frac{x^{k+1}}{1-x^{k+1}}+(2k+4)\frac{x^{k+2}}{1-x^{k+2}}\right)\Bigg|_{x^M}. \quad (4.19)$$

Hence we have

$$f(N,K;\ell_0) = (1+x)^{2M+\ell_0}\left(\frac{1-x^{K+1}}{1-x^{K+2}}\right)^{\ell_0}$$
$$\times\left(1-(2K+2)\frac{x^{K+1}}{1-x^{K+1}}+(2K+4)\frac{x^{K+2}}{1-x^{K+2}}\right)\Bigg|_{x^M}. \quad (4.20)$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-x^{k+1}}{1-x^{k+2}}\right) = \frac{1-x^{k+1}}{x(1-x^{k+2})}\left(-\frac{(k+1)x^{k+1}}{1-x^{k+1}}+\frac{(k+2)x^{k+2}}{1-x^{k+2}}\right)$$

the Cauchy integral is rewritten as

$$\oint_C\frac{\mathrm{d}z}{z^{M+1}}(1+z)^{2M+\ell_0}\left(\frac{1-z^{k+1}}{1-z^{k+2}}\right)^{\ell_0}\left(1-(2k+2)\frac{z^{k+1}}{1-z^{k+1}}+(2k+4)\frac{z^{k+2}}{1-z^{k+2}}\right)$$
$$= \oint_C\frac{\mathrm{d}z}{z^{M+1}}(1+z)^{2M+\ell_0}\left(\frac{1-z^{k+1}}{1-z^{k+2}}\right)^{\ell_0}$$
$$+\oint_C\frac{2\,\mathrm{d}z}{z^{M}}(1+z)^{2M+\ell_0}\left(\frac{1-z^{k+1}}{1-z^{k+2}}\right)^{\ell_0-1}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1-z^{k+1}}{1-z^{k+2}}\right)$$
$$= \oint_C\frac{\mathrm{d}z}{z^{M+1}}(1+z)^{2M+\ell_0}\left(\frac{1-z^{k+1}}{1-z^{k+2}}\right)^{\ell_0}$$
$$+\frac{1}{\ell_0}\oint_C\frac{2\,\mathrm{d}z}{z^{M}}(1+z)^{2M+\ell_0}\frac{\mathrm{d}}{\mathrm{d}z}\left(\left(\frac{1-z^{k+1}}{1-z^{k+2}}\right)^{\ell_0}\right)$$
$$= \oint_C\frac{\mathrm{d}z}{z^{M+1}}(1+z)^{2M+\ell_0}\left(\frac{1-z^{k+1}}{1-z^{k+2}}\right)^{\ell_0}$$
$$-\frac{1}{\ell_0}\oint_C\mathrm{d}z\left(\frac{1-z^{k+1}}{1-z^{k+2}}\right)^{\ell_0}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{2(1+z)^{2M+\ell_0}}{z^M}\right)$$
$$= \frac{\ell_0+2M}{\ell_0}\oint_C\frac{\mathrm{d}z}{z^{M+1}}\left(\frac{1-z^{k+1}}{1-z^{k+2}}\right)^{\ell_0}(1+z)^{2M+\ell_0}$$
$$-\frac{2(2M+\ell_0)}{\ell_0}\oint_C\frac{\mathrm{d}z}{z^{M}}\left(\frac{1-z^{k+1}}{1-z^{k+2}}\right)^{\ell_0}(1+z)^{2M+\ell_0-1}$$

$$= \oint_C \frac{\mathrm{d}z}{z^{M+1}} \left(\frac{\ell_0 + 2M}{\ell_0} - \frac{2(2M+\ell_0)}{\ell_0} \frac{z}{1+z} \right) \left(\frac{1-z^{k+1}}{1-z^{k+2}} \right)^{\ell_0} (1+z)^{2M+\ell_0}$$
$$= \frac{\ell_0 + 2M}{\ell_0} \oint_C \frac{\mathrm{d}z}{z^{M+1}} \left(\frac{1-z^{k+1}}{1-z^{k+2}} \right)^{\ell_0} (1+z)^{2M+\ell_0-1}(1-z). \tag{4.21}$$

Therefore we obtain

Theorem 4.1. *The coefficient $f(N, K; M)$ is given by the Cauchy integral*

$$f(N, K; M) = \frac{N}{2\pi \mathrm{i} \ell_0} \oint_C \frac{\mathrm{d}z}{z^{M+1}} \left(\frac{1-z^{K+1}}{1-z^{K+2}} \right)^{\ell_0} (1+z)^{2M+\ell_0-1}(1-z). \tag{4.22}$$

Here C denotes the contour $|z| = x_0 (<1)$.

We evaluate (4.22) by the method of steepest decent. Let us define

$$f(\zeta) := \left(\frac{1-\mathrm{e}^{(K+1)\zeta}}{1-\mathrm{e}^{(K+2)\zeta}} \right)^{\ell_0} (1+\mathrm{e}^{\zeta})^{2M+\ell_0-1}(1-\mathrm{e}^{\zeta}).$$

Then, by changing variable $z = \mathrm{e}^{\zeta}$,

$$\frac{1}{2\pi \mathrm{i}} \oint_C \frac{\mathrm{d}z}{z^{M+1}} \left(\frac{1-z^{K+1}}{1-z^{K+2}} \right)^{\ell_0} (1+z)^{2M+\ell_0-1}(1-z)$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}v \exp[\log f(\zeta) - M\zeta] \quad (\zeta = u_0 + \mathrm{i}v, \mathrm{e}^{u_0} = x_0).$$

We determine u_0 so that the function $\exp[\log f(\zeta) - M\zeta]$ has the saddle point at $\zeta = u_0$ as

$$[\log f(u_0)]' - M = 0 \tag{4.23}$$
$$[\log f(u_0)]'' > 0. \tag{4.24}$$

Equation (4.23) is rewritten as

$$\ell_0 \left(\frac{-(K+1)\,\mathrm{e}^{(K+1)u_0}}{1-\mathrm{e}^{(K+1)u_0}} - \frac{-(K+2)\,\mathrm{e}^{(K+2)u_0}}{1-\mathrm{e}^{(K+2)u_0}} \right) + (2M+\ell_0) \frac{\mathrm{e}^{u_0}}{1+\mathrm{e}^{u_0}} - \frac{2\,\mathrm{e}^{u_0}}{1-\mathrm{e}^{2u_0}} = M.$$

Since $M = \rho N$ and $\ell_0 = N - 2M = (1-2\rho)N$, we have

$$(1-2\rho)N \left(\frac{-(K+1)\,\mathrm{e}^{(K+1)u_0}}{1-\mathrm{e}^{(K+1)u_0}} - \frac{-(K+2)\,\mathrm{e}^{(K+2)u_0}}{1-\mathrm{e}^{(K+2)u_0}} \right) + N \frac{\mathrm{e}^{u_0}}{1+\mathrm{e}^{u_0}} - \frac{2\,\mathrm{e}^{u_0}}{1-\mathrm{e}^{2u_0}} = \rho N.$$

For $N \gg 1$ and $u_0 < 0$, the third term in the left-hand side of the above equation is negligible. (There is a solution to (4.23) for $u_0 \sim 1 - 0$, but it does not satisfy (4.24).) If we put $\rho =: \frac{t_0}{1+t_0}$ $(0 < t_0 < 1)$ and $\mathrm{e}^{u_0} = t_0 + \varepsilon_K$, we find, at least for sufficiently large K, that there is a unique u_0 which satisfies (4.23) and (4.24) and

$$\varepsilon_K = \left(1 - t_0^2\right)\left((K+1)t_0^{K+1} - (K+2)t_0^{K+2}\right)\left[1 + O\left(t_0^{K+1}\right)\right].$$

Since $\log f(\zeta)$ is written as $\log f(\zeta) = N \log \tilde{f}(\zeta)$ where $\tilde{f}(\zeta)$ depends on N as far as $N \gg 1$ and $\mathrm{Re}[\zeta] < 0$, standard arguments give the asymptotic formula as

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\log f(\zeta)-M\zeta}\,\mathrm{d}v &\simeq \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{(\log f(u_0)-Mu_0)-\frac{1}{2}(\log f(u_0))''v^2}\,\mathrm{d}v \\
&\simeq \frac{1}{2\pi}\mathrm{e}^{(\log f(u_0)-Mu_0)}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}(\log f(u_0))''v^2}\,\mathrm{d}v \\
&= \frac{1}{2\pi}\sqrt{\frac{2\pi}{(\log f(u_0))''}}\,\mathrm{e}^{(\log f(u_0)-Mu_0)} \\
&\sim \frac{1}{\sqrt{2\pi(\log f(u_0))''}}\left(\frac{1-t_0^{k+1}}{1-t_0^{k+2}}\right)^{\ell_0}(1+t_0)^{2M+\ell_0-1}(1-t_0)\frac{1}{t_0^M}.
\end{aligned} \tag{4.25}$$

Here $(\log f(u_0))'' \sim Nt_0/(1+t_0)^2$ $(N \gg 1)$. Hence we have proved

Theorem 4.2. *For sufficiently large K,*

$$f(N,K;M) \sim \frac{N}{\ell_0\sqrt{2\pi t_0 N}}(1-t_0)\frac{(1+t_0)^N}{t_0^M}\left(\frac{1-t_0^{K+1}}{1-t_0^{K+2}}\right)^{\ell_0} \qquad (N \to +\infty) \tag{4.26}$$

where $\ell_0 = N - 2M$, $M = N\rho$ *and* $\rho = \frac{t_0}{1+t_0}$ $(0 < t_0 < 1)$.

Now we discuss the asymptotic behaviour of the fundamental cycle for generic initial states utilizing theorem 4.2. We define the normalized integrated density of states $I_{N;\rho}(K)$ and its derivative $P_{N;\rho}(K)$ as

$$I_{N;\rho}(K) := \frac{f(N,K;M)}{V(N;\rho)} \qquad (M \equiv \rho N) \tag{4.27}$$

$$P_{N;\rho}(K) := I_{N;\rho}(K) - I_{N;\rho}(K-1). \tag{4.28}$$

(Note that $f(N,K;M) = V(N;\rho)$ for $(K \geqslant M)$ is easily confirmed from (4.16).) The function $P_{N;\rho}(K)$ is a normalized density of states the largest solitons of which have length K. From (3.1) and (4.26), we have

$$I_{N;\rho}(K) \simeq \left(\frac{1-t_0^{K+1}}{1-t_0^{K+2}}\right)^{\ell_0} \tag{4.29}$$

$$P_{N;\rho}(K) \simeq -\ell_0(1-t_0)(\log t_0)\frac{t_0^K}{\left(1-t_0^K\right)\left(1-t_0^{K+1}\right)}\left(\frac{1-t_0^K}{1-t_0^{K+1}}\right)^{\ell_0} \tag{4.30}$$

for $K, N \gg 1$. The function $P_{N;\rho}(K)$ has one sharp peak at

$$\begin{aligned}
K_{\max} &\simeq \frac{\log(\ell_0(1-t_0))}{-\log t_0} \\
&\simeq \frac{\log N}{-\log t_0}.
\end{aligned} \tag{4.31}$$

To know the width of this peak, we define $K_\pm(\epsilon)$ for a given small positive number $(0 < \epsilon \ll 1)$ as

$$I_{N;\rho}(K_-) = \epsilon \qquad I_{N;\rho}(K_+) = 1 - \epsilon. \tag{4.32}$$

Precisely speaking, K_- (K_+) $\in \mathbb{Z}_+$ is the integer that minimizes the value $|I_{N;\rho}(K) - \epsilon|$ ($|I_{N;\rho}(K) - (1-\epsilon)|$). From (4.29), they are evaluated as

$$K_-(\epsilon) \simeq \frac{\log(\ell_0(1-t_0))}{-\log t_0} - \frac{\log(-\log\epsilon)}{-\log t_0}$$
$$\simeq K_{\max} - \frac{\log(-\log\epsilon)}{-\log t_0} \tag{4.33}$$

$$K_+(\epsilon) \simeq \frac{\log(\ell_0(1-t_0))}{-\log t_0} + \frac{-\log(-\log(1-\epsilon))}{-\log t_0}$$
$$\simeq K_{\max} + \frac{-\log(-\log(1-\epsilon))}{-\log t_0}. \tag{4.34}$$

Thus the width is very narrow—even for $\epsilon = 1/N$, $K_+ - K_- \sim \log N$. Therefore, we found that most states have the largest soliton whose length is of the order of $\frac{\log N}{-\log t_0}$.

Let $V_K(N;\rho)$ be the number of initial states the largest solitons of which have length K. From the definition of $K_+(\epsilon)$,

$$\frac{V_{K_+(\epsilon)}(N;\rho)}{V(N;\rho)} = 1-\epsilon. \tag{4.35}$$

On the other hand, the fundamental cycle of states which consist of solitons which have length K or less satisfies

$$T < N^K. \tag{4.36}$$

From (4.31), $\exists C \in \mathbb{R}_+$ such that

$$K_+(\epsilon) \leqslant \frac{\log N}{-\log t_0} + C\frac{-\log\epsilon}{-\log t_0}. \tag{4.37}$$

Then (4.36) yields

$$T < \exp[K_+(\epsilon)\log N] \leqslant \exp\left[\frac{(\log N)^2}{-\log t_0}\left(1 + C\frac{-\log\epsilon}{\log N}\right)\right]. \tag{4.38}$$

Let δ be an arbitrary positive number. For $\forall\epsilon > 0$, we denote by $\bar{V}_\delta(N;\rho)$ the number of fundamental cycles which does not exceed $\exp\left[\frac{(1+\delta)(\log N)^2}{-\log t_0}\right]$. Then, from (4.37), for any positive integer N which satisfies

$$\delta\log N > -C\log\epsilon$$

we have

$$K_+ < \frac{\log N}{-\log t_0}(1+\delta).$$

Therefore, for arbitrary $\epsilon > 0$, if $N > \exp[-(C\log\epsilon)/\delta]$, we have

$$1-\epsilon \leqslant \frac{\bar{V}_\delta(N;\rho)}{V(N;\rho)} < 1.$$

In conclusion, we have proved

Theorem 4.3. *Let $\bar{V}_\delta(N;\rho)$ be the number of initial states which have a fundamental cycle less than* $\exp\left[\frac{(1+\delta)(\log N)^2}{-\log t_0}\right]$. *Then, for $\forall\delta > 0$,*

$$\lim_{N\to\infty}\frac{\bar{V}_\delta(N;\rho)}{V(N;\rho)} = 1. \tag{4.39}$$

5. Concluding remarks

We have investigated the integrability of PBBS in terms of the asymptotic behaviour of its fundamental cycles. As a dynamical system, PBBS is shown to have no ergodicity in the sense that a trajectory does not visit most of the states in the phase space. Although the maximum fundamental cycle $T_{\max} \lesssim e^{\sqrt{N}}$ (theorem 3.1), a generic state has fundamental cycle $T \lesssim e^{(\log N)^2}$ (theorem 4.3). To obtain a sharper estimation, we may have to invoke some number theoretical techniques, which is a problem we wish to address in the future.

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